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On Π_2 theories of hp-T degrees of low sets

Kunimasa Aoki

Department of Mathematics, Nagoya University, Nagoya 464-01, Japan

Juichi Shinoda

Department of Mathematics, College of General Education, Nagoya University, Nagoya 464-01, Japan

Teruko Tsuda

Division of Intelligence Science, Graduate School of Science and Technology, Kobe University, Nada-ku, Kobe 657, Japan

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Abstract

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If I is a set of Δ_2^0 low sets which is closed under recursive join and downward closed under $\leqslant_T$, then it is shown that the Π_2 theory of the honest polynomial time Turing degrees of the sets in I is the same as the Π_2 theory of the polynomial time Turing degrees of the recursive sets, and so that it is decidable.

1. Introduction

The honest polynomial-time Turing (hp-T) reducibility, $\leqslant_T^{hp}$, was first introduced by Homer (e.g. [6]). It has been known that the structural properties of the hp-T degrees of the recursive sets and those of the Δ_2^0 sets are closely related to the P=?NP problem.

Homer [6, 7] and Homer and Long [8] proved assuming P=NP that there is a Δ_3^0 set which is $\leqslant_T^{hp}$ minimal, and then Ambos-Spies [2] improved their results by

Correspondence to: T. Tsuda, 5-11-901, Tanaka-cho, Nishinomiya-shi, 662, Japan. Email: tsuda@kurims.kyoto-u.ac.jp.

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showing under the same assumption that there is a minimal one which is recursively enumerable. It is conjectured that the existence of such sets is equivalent to $P=NP$ (see [6, 8]). Downey [5], on the other hand, recently proved that there is no $\leqslant_T^{hp}$-minimal low set, where a set A is called *low* if $A' \equiv_T \emptyset'$. Furthermore, in [4], we have extended Downey's result and shown that the strong minimal pair theorem holds for the hp-T degrees of the low sets, which implies the density of the hp-T degrees of the low sets.

Theorem 1.1 (The strong minimal pair theorem, Aoki et al. [4]). *Given low sets A and B with $A <_T^{hp} B$, there are two sets C and D such that $A <_T^{hp} C <_T^{hp} B$, $A <_T^{hp} D <_T^{hp} B$ and*

$$\deg_T^{hp}(C) \wedge \deg_T^{hp}(D) = \deg_T^{hp}(A).$$

Ambos-Spies [2] also proved that if $P=NP$ then the theory of $\boldsymbol{R}_T^{hp}$ is different from the theory of $\boldsymbol{R}_T^{p}$, where $\boldsymbol{R}_T^{hp}$ and $\boldsymbol{R}_T^{p}$ are the ordered structures consisting of the hp-T degrees and the p-T degrees of the recursive sets, respectively: for any recursive set $A \notin P$, there is a recursive set $B \leqslant_T^p A$ such that $B \notin P$ and B does not $\leqslant_T^p$-help A, while if $P=NP$ then there is a recursive set $A \notin P$ which is $\leqslant_T^{hp}$-helped by all of its $\leqslant_T^{hp}$-predecessors which are not in P. If we drop the assumption $P=NP$, then it is not known whether the theory of $\boldsymbol{R}_T^{hp}$ is different from the theory of $\boldsymbol{R}_T^{p}$ (see [2]). In this paper, we shall apply the method of Shore and Slaman [11] for deciding the Π_2 theory of $\boldsymbol{R}_T^{p}$, i.e., the $\forall\exists$-sentences true in $\boldsymbol{R}_T^{p}$, to the Π_2 theories of ideals of hp-T degrees of low sets. Suppose $\boldsymbol{I}$ is a nonempty collection of low sets and satisfies the following conditions:

(1) if $A \in \boldsymbol{I}$ and B is recursive in A, then $B \in \boldsymbol{I}$,

(2) if $A, B \in \boldsymbol{I}$, then $A \oplus B \in \boldsymbol{I}$,

where $A \oplus B$ denotes the recursive join of A and B. Then, $I = \{\deg_T^{hp}(A) \colon A \in \boldsymbol{I}\}$ forms an ideal of $\mathscr{D}_T^{hp}$, the hp-T degrees of all sets. In this paper, we shall prove the following theorem.

Theorem 1.2. *Consider any ideal I described above. The Π_2 theories of $\langle I, \leqslant_T^{hp}\rangle$ and $\boldsymbol{R}_T^{p}$ are identical. In particular, as to the Π_2 sentences, the ordered structures $\boldsymbol{R}_T^{hp}$ and $\boldsymbol{R}_T^{p}$ cannot be distinguished.*

Our proof of the theorem depends on the logical analysis of the Π_2 sentences given by Shore and Slaman [11] (see also [10]). They have shown that the Π_2 theory of $\boldsymbol{R}_T^{p}$ is decidable by proving two key theorems: the lattice embedding theorem and the extension theorem of embeddings. Their analysis shows that for any upper semilattice if these two theorems are once established then the Π_2 theory of the upper semilattice is decidable; furthermore, the same decision procedure as that for the Π_2 theory of $\boldsymbol{R}_T^{p}$ is applied to this theory, and these two theories coincide. Thus, to prove our theorem, it is sufficient to demonstrate the lattice embedding theorem and the extension theorem for $\langle I, \leqslant_T^{hp}\rangle$.

This paper is organized as follows. In Section 2, we introduce the basic notions of the hp-T reducibility. In particular, the lattice embedding theorem for $\langle I, \leqslant_T^{hp}\rangle$ is deduced from the same theorem for $\boldsymbol{R}_T^p$ as Ambos-Spies and Yang [3] noticed. In Section 3, we review the extension theorem in the form given in [11]. We list the requirements needed to extend a given embedding and, in Section 4, we shall construct an extension which satisfies these requirements. Finally, in Section 5, we give a remark on the density of the p-T degrees of arbitrary sets and some open questions.

Throughout this paper, we fix a collection $\boldsymbol{I}$ of low sets which satisfies the conditions (1) and (2) above.

2. Honest polynomial-time Turing degrees

Let $\Sigma = \{0, 1\}$, and let Σ^* be the set of finite strings over Σ with natural ordering, whose elements are denoted by $x, y, z, \ldots$. We use $A, B, C, \ldots$ to denote subsets of Σ^*. Let $|x|$ denote the length of x and $\langle\, ,\rangle$ be a p-time invertible bijection from $\Sigma^* \times \Sigma^*$ to Σ^*. We may assume that $|x|, |y| \leqslant |\langle x, y\rangle|$. An oracle Turing machine (OTM) Φ is *polynomially honest* if there is a polynomial p such that on input x, Φ queries the oracle only on strings y with $|x| \leqslant p(|y|)$. A is *p-T reducible* to B, $A \leqslant_T^p B$, if there is a polynomial-time bounded deterministic OTM Φ such that $A = \Phi(B)$; furthermore, if Φ is polynomially honest, then A is said to be *hp-T reducible* to B, $A \leqslant_T^{hp} B$. Note that if $A \leqslant_T^{hp} B$, the oracle B can be queried only on strings y whose lengths are polynomially related to $|x|$, i.e., $|y| \leqslant p(|x|)$ and $|x| \leqslant p(|y|)$ for some polynomial p. This contrasts with the p-T reducibility, where $|y|$ is polynomially bounded in $|x|$ but, in general, not vice versa.

A and B have the same hp-T *degrees* if A $\leqslant_T^{hp} B$ and $B \leqslant_T^{hp} A$. The hp-T degree of A is denoted by $\deg_T^{hp}(A)$. The p-T degree of A, $\deg_T^p(A)$, is defined in the same way. The recursive join $A \oplus B$ of sets A and B is defined by

$$A \oplus B = \{0x\colon x \in A\} \cup \{1x\colon x \in B\}.$$

The hp-T degree of $A \oplus B$ is the least upper bound of $\deg_T^{hp}(A)$ and $\deg_T^{hp}(B)$.

$\mathrm{Cyl}(A) = A \times \Sigma^*$ is called the *p-cylindrification* of A. We say that A is a *p-cylinder* if there is a p-cylindrification B to which A is p-isomorphic, i.e., there is a one to one onto p-time computable function $f\colon \Sigma^* \to \Sigma^*$ whose inverse is also p-time-computable such that $A = f(B)$. The following lemma is used to translate the results on the p-T degrees to those on the hp-T degrees.

Lemma 2.1 (Ambos-Spies [2, 3]). *Let A, B, and C be any sets.*

(i) *If $A \leqslant_T^{hp} B$ then $A \leqslant_T^p B$.*
(ii) $A \equiv_T^p \mathrm{Cyl}(A)$.
(iii) *If A is a p-cylinder then $B \leqslant_T^{hp} A$ iff $B \leqslant_T^p A$.*
(iv) $\deg_T^p(\mathrm{Cyl}(A)) \vee \deg_T^p(\mathrm{Cyl}(B)) = \deg_T^p(\mathrm{Cyl}(A \oplus B))$.

Moreover, if $\deg_{\mathrm{T}}^{\mathrm{p}}(A) \wedge \deg_{\mathrm{T}}^{\mathrm{p}}(B) = \deg_{\mathrm{T}}^{\mathrm{p}}(C)$, *then*

$$\deg_{\mathrm{T}}^{\mathrm{hp}}(\mathrm{Cyl}(A)) \wedge \deg_{\mathrm{T}}^{\mathrm{hp}}(\mathrm{Cyl}(B)) = \deg_{\mathrm{T}}^{\mathrm{hp}}(\mathrm{Cyl}(C)).$$

Proof. (i) and (ii) are trivial.

For (iii), we may assume that $A = \mathrm{Cyl}(C)$ for some set C. For nontrivial "if" part, assume $B \leqslant_{\mathrm{T}}^{\mathrm{p}} A$. Since A is p-T reducible to C, so is B. Let Φ be a p-time OTM such that $B = \Phi(C)$. In the computation of $\Phi(C, x)$ on input x, we replace each query "$y \in C$?" with the equivalent query "$\langle y, x\rangle \in A$?". Note that $|x| \leqslant |\langle y, x\rangle|$ and $|\langle y, x\rangle|$ is polynomially bounded by $|x|$. In this way, we can obtain an hp-T reduction of B to A.

Part (iv) is proved immediately by (i) and (iii). □

Corollary 2.2 (the lattice embedding theorem). *Every finite lattice is embeddable into* $\boldsymbol{R}_{\mathrm{T}}^{\mathrm{hp}}$ *preserving* $\vee$ *and* $\wedge$, *and hence it is embeddable into* $\langle I, \leqslant_{\mathrm{T}}^{\mathrm{hp}}\rangle$.

Proof. The above lemma shows that the map $\deg_{\mathrm{T}}^{\mathrm{p}}(A) \mapsto \deg_{\mathrm{T}}^{\mathrm{hp}}(\mathrm{Cyl}(A))$ gives an isomorphism from $\boldsymbol{R}_{\mathrm{T}}^{\mathrm{p}}$ into $\boldsymbol{R}_{\mathrm{T}}^{\mathrm{hp}}$. □

3. Extension of an embedding

In this section, let $\mathscr{I}$ denote the ordered structure $\langle I, \leqslant_{\mathrm{T}}^{\mathrm{hp}}\rangle$, where

$$I = \{\deg_{\mathrm{T}}^{\mathrm{hp}}(A) \colon A \in \boldsymbol{I}\}.$$

Then, $\mathscr{I}$ is an upper semilattice since $\boldsymbol{I}$ is closed under $\oplus$.

Lemma 3.1. *Given* $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in I$, *there is a* $\boldsymbol{b} \in I$ *such that* $\boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_n < \boldsymbol{b}$.

Proof. Let $\boldsymbol{a} = \boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_n$ and suppose $\boldsymbol{a} = \deg_{\mathrm{T}}^{\mathrm{hp}}(A)$ with $A \in \boldsymbol{I}$. By a simple diagonal argument, we can construct a set B which is recursive in A but not hp-T reducible to A. Then $A \oplus B \in \boldsymbol{I}$ and $A <_{\mathrm{T}}^{\mathrm{hp}} A \oplus B$. □

Suppose $M = \{0, a_1, \ldots, a_{n-1}, b_1, \ldots, b_m\}$ is a finite partially ordered set with order $\preceq$, where 0 is the least member, and suppose $L = \{0, a_1, \ldots, a_{n-1}\}$ forms a lattice with respect to $\preceq$. $a_i \vee a_j$ denotes the least upper bound of $\{a_i, a_j\}$ in the lattice L and $a_i \wedge a_j$ the greatest lower bound. Suppose, furthermore, $b_1, \ldots, b_m$ satisfies the following conditions:

(1) if $a_i, a_j \preceq b_k$, then $a_i \vee a_j \preceq b_k$;

(2) if $b_k \preceq a_i, a_j$, then $b_k \preceq a_i \wedge a_j$.

In these circumstances, the extension theorem for $\mathscr{I}$ is stated as follows (see [11]).

Theorem 3.2. *Let* M *and* L *be given as above. If* $f\colon L \to \mathscr{I}$ *is an order isomorphism of* L *into* $\mathscr{I}$ *such that* $f(0) = \deg_{\mathrm{T}}^{\mathrm{hp}}(\emptyset)$, *then there exists an order isomorphism of* M *into* $\mathscr{I}$ *which extends* f.

Hereafter, we set $a_0=0$. Suppose $f\colon L\to\mathscr{I}$ is a given embedding, and for each $i\,(0\leqslant i<n)$, let A_i be an element of $\boldsymbol{I}$ such that $f(a_i)=\deg_{\mathrm{T}}^{\mathrm{hp}}(A_i)$. Given b_j, there is, by (1), a greatest a_i below b_j, and if there is an a_i such that $b_j\preceq a_i$ then, by (2), there is a smallest one. We use $a_{G(j)}$ and $a_{S(j)}$ to denote these elements, respectively,

$$a_{G(j)}=\bigvee\{a_i\colon a_i\preceq b_j\},$$

$$a_{S(j)}=\bigwedge\{a_i\colon b_j\preceq a_i\}.$$

When there is no a_i such that $b_j\preceq a_i$, we set $S(j)=n$. By Lemma 3.1, there is an $A_n\in\boldsymbol{I}$ such that $A_0\oplus\cdots\oplus A_{n-1}<_{\mathrm{T}}^{\mathrm{hp}}A_n$. We add a new element a_n to M, and extend $\preceq$ and f on $M\cup\{a_n\}$ by putting a_n above all elements of M and $f(a_n)=\deg_{\mathrm{T}}^{\mathrm{hp}}(A_n)$. As in [11], we shall construct sets $C_k\leqslant_{\mathrm{T}}^{\mathrm{hp}}A_{S(k)}$ and then set $B_k=A_{G(k)}\oplus\bigoplus_{b_j\preceq b_k}C_j$. The desired extension will be defined by setting $f(b_j)=\deg_{\mathrm{T}}^{\mathrm{hp}}(B_j)$. $B_k\in\boldsymbol{I}$ follows from the closure property of $\boldsymbol{I}$. It is clear from the definition that if $b_j\preceq b_k$ then $B_j\leqslant_{\mathrm{T}}^{\mathrm{hp}}B_k$. Therefore, to ensure that the extension f so defined is an order isomorphism on $M\cup\{a_n\}$, the following conditions must be satisfied:

R_1: if $a_i\npreceq b_j$, then $A_i\nleqslant_{\mathrm{T}}^{\mathrm{hp}}B_j$;

R_2: if $b_i\npreceq b_j$, then $B_i\nleqslant_{\mathrm{T}}^{\mathrm{hp}}B_j$;

R_3: if $b_i\npreceq a_j$, then $B_i\nleqslant_{\mathrm{T}}^{\mathrm{hp}}A_j$.

Let $\{\Phi_e,p_e\}_e$ be a recursive enumeration of the polynomially honest OTMs. Without loss of generality, we may assume each a_j is equal to some b_k in M since we may add new elements $b_{m+1},\ldots,b_{m+n}$ to M and extend the diagram of M by claiming that each b_{m+j} is equal to a_j. Then the requirements R_3 can be replaced by the new requirement of type R_2:

$$\text{if } b_i\npreceq b_{m+j}, \text{ then } B_i\nleqslant_{\mathrm{T}}^{\mathrm{hp}}B_{m+j}.$$

Thus, hereafter, we consider only the requirements of forms R_1 and R_2. To ensure the above conditions, it is sufficient to meet all requirements of the following forms:

$R(e,i,j,1)$: $A_i\neq\Phi_e(B_j)$, where $a_i\npreceq b_j$;

$R(e,i,j,2)$: $C_i\neq\Phi_e(B_j)$, where $b_i\npreceq b_j$.

In the next section, we shall construct $C_1,\ldots,C_m\in\boldsymbol{I}$ to satisfy all of the above requirements.

4. Construction of an extension

Before giving the details of the construction, we first consider the simplest case where all A_i's are recursive. We order the requirements with priority. A requirement $R(e_1,i_1,j_1,k_1)$ has stronger priority than a requirement $R(e_2,i_2,j_2,k_2)$ if $(e_1,i_1,j_1,k_1)<(e_2,i_2,j_2,k_2)$ in the lexicographic order. We define a strictly increasing sequence $\{l_n\}_n$ and sets $C_1,\ldots,C_m$ by recursion as follows.

Let $l_0=0$.

Suppose l_n and $C_k\restriction\{z\colon|z|<l_n\}$ are defined. We define l_{n+1} and $C_k\restriction\{z\colon|z|<l_{n+1}\}$ by cases.

Case 1: The nth requirement has the form $R(e, i, j, 1)$. Then we see that $A_i \not\leqslant_T^{hp} A_{G(j)}$ since $a_i \not\leqslant b_j$, and thus there is an x such that $p_e(l_n) \leqslant |x|$ and $A_i(x) \neq \Phi_e(A_{G(j)}, x)$. Take the least such x. Note that if Φ_e queries $A_{G(j)}$ on y in the computation of $\Phi_e(A_{G(j)}, x)$, then $l_n \leqslant |y| \leqslant p_e(|x|)$. Let $l_{n+1} = l_n$ plus the total number of steps needed to find the x and verify the inequality $A_i(x) \neq \Phi_e(A_{G(j)}, x)$, in which the steps needed to compute $A_i(z)$ for z less than or equal to x must be included. We set, for every k, $C_k(z) = 0$, so $B_j = A_{G(j)}$ on the interval $\{z\colon l_n \leqslant |z| < l_{n+1}\}$. Then we have $A_i(x) \neq \Phi_e(B_j, x)$ and thus the requirement $R(e, i, j, 1)$ is satisfied.

Case 2: The nth requirement is $R(e, i, j, 2)$. In this case, we have $A_{S(i)} \not\leqslant_T^{hp} A_{G(j)}$ since $b_i \not\leqslant b_j$. Thus, there is an x such that $p_e(l_n) \leqslant |x|$ and $A_{S(i)}(x) \neq \Phi_e(A_{G(j)}, x)$. We define x and l_{n+1} as in case 1. We set $C_i(z) = A_{S(i)}(z)$ and for each k with $k \neq i$ set $C_k(z) = 0$ on the interval $\{z\colon l_n \leqslant |z| < l_{n+1}\}$. Then $B_j = A_{G(j)}$ on this interval. We have $C_i(x) \neq \Phi_e(B_j, x)$ and thus the requirement $R(e, i, j, 2)$ is satisfied.

To show that C_k so-defined satisfies $C_k \leqslant_T^{hp} A_{S(k)}$, let x be an arbitrary element of Σ^*. First, find the unique n such that $l_n \leqslant |x| < l_{n+1}$. This can be done by performing the above construction in $|x|$ steps. Then if the nth requirement is $R(e, i, j, 2)$, then $C_k(x) = A_{S(i)}(x)$ or $C_k(x) = 0$ depending on whether $k = i$ or not. Thus, $C_k(x)$ is computed from $A_{S(k)}$ in polynomial time. Another case is treated similarly. It is obvious that this reduction of C_k to $A_{S(k)}$ is honest.

When some A_i is not recursive, the sequence $\{l_n\}_n$ defined above is not necessarily recursive, which gives rise to an intrinsic obstacle to verify that $C_k \leqslant_T^{hp} A_{S(k)}$. Here we are assuming all A_is are Δ_2^0; thus, by the limit lemma [12, III.3.3], there exists a recursive function $f_i(x, s)$ such that $f_i(x, s) \leqslant 1$ and

$$A_i(x) = \lim_s f_i(x, s).$$

Let $A_{i,s}(x) = f_i(x, s)$. In the above construction, we can make use of $A_{i,s}$ in place of A_i. For example, in case 1, first find an s and x such that $A_{i,s}(x) \neq \Phi_e(A_{G(j),s}, x)$, then define l_{n+1} and $C_k \upharpoonright \{z\colon l_n \leqslant |z| < l_{n+1}\}$ as before. The requirement $R(e, i, j, 1)$ would be temporarily satisfied. However, it might happen at later $t > s$ that $A_{i,t}(x) = \Phi_e(A_{G(j),t}, x)$ and thus the requirement $R(e, i, j, 1)$ would be injured. Then we must attack $R(e, i, j, 1)$ again. But, it is not clear that each $R(e, i, j, k)$ is injured only finitely often. Here we use the lowness of A_i to cope with this difficulty (see [4] or [5]).

Let p_e^x denote the interval $\{y \in \Sigma^*\colon p_e^{-1}(|x|) \leqslant |y| \leqslant p_e(|x|)\}$ determined by polynomial p_e, and let $\{\sigma_n\}_n$ denote a recursive enumeration of the finite functions σ such that $\mathrm{dom}(\sigma) = p_e^x$ for some e, x and $\mathrm{rng}(\sigma) \subseteq \{0, 1\}$. Note that if $\mathrm{dom}(\sigma) \supseteq p_e^x$ then it is safe to write $\Phi_e(\sigma, x)$ since on input x, Φ_e queries the oracle only on the elements of p_e^x. Define $H_{i,j}$ and $\hat{H}_{i,j}$ by

$$H_{i,j} = \{e'\colon (\exists \langle x, n\rangle \in W_{e'})[x \in A_i \ \&\ \sigma_n = A_j \upharpoonright p_e^x \quad \text{for some } e]\},$$

$$\hat{H}_{i,j} = \{e'\colon (\exists \langle x, n\rangle \in W_{e'})[x \notin A_i \ \&\ \sigma_n = A_j \upharpoonright p_e^x \quad \text{for some } e]\},$$

where W_e is the eth recursively enumerable set. Then since $A_i \oplus A_j$ is low, $H_{i,j}$ and $\hat{H}_{i,j}$ are Δ_2^0. Using the limit lemma, there are recursive functions $h_{i,j}$ and $\hat{h}_{i,j}$ such that

$h_{i,j}(x,s) \leqslant 1$, $\hat{h}_{i,j}(x,s) \leqslant 1$ and

$$H_{i,j}(x) = \lim_s h_{i,j}(x,s) \quad \text{and} \quad \hat{H}_{i,j}(x) = \lim_s \hat{h}_{i,j}(x,s).$$

During the construction below, we shall define a strictly increasing sequence $\{l_N\}_N$ and auxiliary functions $l(s)$, $t(N)$ and $r(s)$. When $l(s) = l_N$, the requirement $R(t(N))$ will be attacked at stage $s+1$. We use c as a global counter. At the beginning of the construction, c is set to 0 and incremented by one at every step of the machine carrying out the construction other than those needed to increment the counter. Then, for each x, we define C_i by

$$C_i(x) = \begin{cases} A_{S(i)}(x) & \text{if } t(N) = (e, i, j, 2) \\ 0 & \text{otherwise,} \end{cases}$$

where N is the unique number such that $l_N \leqslant |x| < l_{N+1}$.

We shall also build recursive sequences $\{V_s(e,i,j,k)\}_s$ and $\{\hat{V}_s(e,i,j,k)\}_s$ of finite sets during the construction. Let $V(e,i,j,k) = \bigcup_s V_s(e,i,j,k)$ and $\hat{V}(e,i,j,k) = \bigcup_s \hat{V}_s(e,i,j,k)$. Then $V(e,i,j,k)$ and $\hat{V}(e,i,j,k)$ are recursively enumerable. By the recursion theorem we may assume that we have in advance an index $\theta(e,i,j,k)$ of $V(e,i,j,k)$ and an index $\hat{\theta}(e,i,j,k)$ of $\hat{V}(e,i,j,k)$, where θ and $\hat{\theta}$ are recursive. Thus, $W_{\theta(e,i,j,k)} = V(e,i,j,k)$ and $W_{\hat{\theta}(e,i,j,k)} = \hat{V}(e,i,j,k)$. The sets $V(e,i,j,k)$ and $\hat{V}(e,i,j,k)$ will be so constructed that

$$V(e,i,j,1) \subseteq \{\langle x,n\rangle : (\exists s)[A_{i,s}(x) = 1 \ \&\ \Phi_e(\sigma_n, x) = 0 \ \&\ \sigma_n = A_{G(j),s} \upharpoonright p_e^x]\},$$

$$\hat{V}(e,i,j,1) \subseteq \{\langle x,n\rangle : (\exists s)[A_{i,s}(x) = 0 \ \&\ \Phi_e(\sigma_n, x) = 1 \ \&\ \sigma_n = A_{G(j),s} \upharpoonright p_e^x]\},$$

$$V(e,i,j,2) \subseteq \{\langle x,n\rangle : (\exists s)[A_{S(i),s}(x) = 1 \ \&\ \Phi_e(\sigma_n, x) = 0 \ \&\ \sigma_n = A_{G(j),s} \upharpoonright p_e^x]\},$$

$$\hat{V}(e,i,j,2) \subseteq \{\langle x,n\rangle : (\exists s)[A_{S(i),s}(x) = 0 \ \&\ \Phi_e(\sigma_n, x) = 1 \ \&\ \sigma_n = A_{G(j),s} \upharpoonright p_e^x]\}.$$

At each stage, the construction takes one of the three phases, *testing phase*, *waiting phase* and *checking phase*. If the construction is in testing phase, then we will attack a requirement in *LIST*, the list of uncertified requirements, with the highest priority. In waiting phase, we do nothing but only increase the counter. If a stage is in checking phase, we will check whether the requirements which have already been certified are still certified at the stage. We now give a formal construction.

Construction

Stage 0

We set $c = 0$. Set $l(0) = l_0 = 0$ and $V_0(e,i,j,k) = \hat{V}_0(e,i,j,k) = \emptyset$ for every (e,i,j,k). Declare all requirements to be uncertified. Thus, *LIST* consists of all requirements. Set $r(0) = t(0) = (e_0, i_0, j_0, k_0)$, where $R(e_0, i_0, j_0, k_0)$ is the first element of *LIST*. Declare 0 to be in testing phase.

Stage $s+1$

Suppose $l(s)=l_N$.

Case 1: s is in testing phase.

Suppose $R(t(N))$ is the requirement of the highest priority in *LIST*.

Case 1.1: $t(N)=(e, i, j, 1)$.

Case 1.1.1: There exists an x with $p_e(l_N)\leqslant|x|<s+1$ such that one of the following holds:

(1) $A_{i,s+1}(x)=1$ and $\Phi_e(A_{G(j),s+1}, x)=0$,

(2) $A_{i,s+1}(x)=0$ and $\Phi_e(A_{G(j),s+1}, x)=1$.

Then, take the least such x. If x satisfies (1) then set

$$V_{s+1}(e, i, j, 1)=V_s(e, i, j, 1)\cup\{\langle x, n\rangle\},$$

otherwise set

$$\hat{V}_{s+1}(e, i, j, 1)=\hat{V}_s(e, i, j, 1)\cup\{\langle x, n\rangle\},$$

where $\sigma_n=A_{G(j),s+1}\restriction p_e^x$.

Case 1.1.1.1: One of the following holds:

(i) $h_{i,G(j)}(\theta(e, i, j, 1), s+1)=1$ and
$(\exists\langle x, n\rangle\in V_{s+1}(e, i, j, 1))\ [A_{i,s+1}(x)=1\ \&\ \sigma_n=A_{G(j),s+1}\restriction p_e^x]$,

(ii) $\hat{h}_{i,G(j)}(\hat{\theta}(e, i, j, 1), s+1)=1$ and
$(\exists\langle x, n\rangle\in \hat{V}_{s+1}(e, i, j, 1))\ [A_{i,s+1}(x)=0\ \&\ \sigma_n=A_{G(j),s+1}\restriction p_e^x]$.

In this case, we say that the requirement $R(e, i, j, 1)$ is *certified*. Set l_{N+1} to the current value of the counter c. Remove $R(e, i, j, 1)$ from *LIST*. Declare $s+1$ to be in waiting phase. Set $l(s+1)=l(s)$.

Case 1.1.1.2: Otherwise.

Set $l(s+1)=l(s)$. Do nothing else.

Case 1.1.2: Otherwise.

Set $l(s+1)=l(s)$. Do nothing else.

Case 1.2: $t(N)=(e, i, j, 2)$.

Case 1.2.1: There exists an x with $p_e(l_N)\leqslant|x|<s+1$ such that one of the following holds:

(1) $A_{S(i),s+1}(x)=1$ and $\Phi_e(A_{G(j),s+1}, x)=0$,

(2) $A_{S(i),s+1}(x)=0$ and $\Phi_e(A_{G(j),s+1}, x)=1$.

Take the least such x. If x satisfies (1) then set

$$V_{s+1}(e, i, j, 2)=V_s(e, i, j, 2)\cup\{\langle x, n\rangle\},$$

otherwise set

$$\hat{V}_{s+1}(e, i, j, 2)=\hat{V}_s(e, i, j, 2)\cup\{\langle x, n\rangle\},$$

where $\sigma_n=A_{G(j),s+1}\restriction p_e^x$.

Case 1.2.1.1: One of the following holds:

(i) $h_{S(i),G(j)}(\theta(e, i, j, 2), s+1)=1$ and
$(\exists\langle x, n\rangle\in V_{s+1}(e, i, j, 2))\ [A_{S(i),s+1}(x)=1\ \&\ \sigma_n=A_{G(j),s+1}\restriction p_e^x]$,

(ii) $\hat{h}_{S(i),G(j)}(\hat{\theta}(e,i,j,2),s+1)=1$ and
$(\exists\langle x,n\rangle\in\hat{V}_{s+1}(e,i,j,2))\ [A_{S(i),s+1}(x)=0\ \&\ \sigma_n=A_{G(j),s+1}\upharpoonright p_e^x]$.
In this case, we say that the requirement $R(e,i,j,2)$ is *certified*. Then, we set l_{N+1} to the current value of the counter c. Remove $R(e,i,j,2)$ from *LIST* and declare $s+1$ to be in waiting phase. Set $l(s+1)=l(s)$.

Case 1.2.1.2: Otherwise.
Set $l(s+1)=l(s)$. Do nothing else.

Case 1.2.2: Otherwise.
Set $l(s+1)=l(s)$. Do nothing else.

Case 2: s is in waiting phase.

Case 2.1: $s=l_{N+1}$.
Set $l(s)=l_{N+1}$. Declare $s+1$ to be in checking phase.

Case 2.2: Otherwise.
Set $l(s+1)=l(s)$. Do nothing else.

Case 3: s is in checking phase.

Case 3.1: There exists a requirement $R(r)$ with $r\leqslant r(s)$ such that one of the following holds:

(1) $r=(e,i,j,1)$ for some e,i,j and the following (1.1) and (1.2) hold:

(1.1) $h_{i,G(j)}(\theta(r),s)=0$ and
$$(\forall\langle x,n\rangle\in V_s(r))\ [A_{i,s}(x)=0\ \vee\ \sigma_n\neq A_{G(j),s}\upharpoonright p_e^x],$$

(1.2) $\hat{h}_{i,G(j)}(\hat{\theta}(r),s)=0$ and
$$(\forall\langle x,n\rangle\in\hat{V}_s(r))\ [A_{i,s}(x)=1\ \vee\ \sigma_n\neq A_{G(j),s}\upharpoonright p_e^x].$$

(2) $r=(e,i,j,2)$ for some e,i,j and the following (2.1) and (2.2) hold:

(2.1) $h_{S(i),G(j)}(\theta(r),s)=0$ and
$$(\forall\langle x,n\rangle\in V_s(r))\ [A_{S(i),s}(x)=0\ \vee\ \sigma_n\neq A_{G(j),s}\upharpoonright p_e^x],$$

(2.2) $\hat{h}_{S(i),G(j)}(\hat{\theta}(r),s)=0$ and
$$(\forall\langle x,n\rangle\in\hat{V}_s(r))\ [A_{S(i),s}(x)=1\ \vee\ \sigma_n\neq A_{G(j),s}\upharpoonright p_e^x].$$

In this case, we say that the requirement $R(r)$ is *injured*. Take the least such r and put $R(r)$ into *LIST*. Thus, the requirement $R(r)$ is *uncertified* at this stage. Declare $s+1$ to be in testing phase. Set $l(s+1)=l(s)$.

Case 3.2: Otherwise.
Set $r(s+1)$ so that $R(r(s+1))$ is the next requirement after $R(r(s))$. Declare $s+1$ to be in testing phase. Set $l(s+1)=l(s)$.

End

Verification

Suppose x is an arbitrary element of Σ^*. $C_i(x)$ is calculated from $A_{S(i)}$ as follows. First, by carrying out the construction in $|x|$ steps, we can find the unique N such that $l_N\leqslant|x|<l_{N+1}$ and calculate the value of $t(N)$. Then if $t(N)=(e,i,j,2)$ for some e and

j then $C_i(x) = A_{S(i)}(x)$; otherwise, $C_i(x) = 0$. It is easy to see that this gives an hp-T reduction of C_i to $A_{S(i)}$.

To see the correctness of the construction, we show the following claims.

Claim 4.1. *If the construction is in testing phase at stage s, then it turns into waiting phase at some stage after s.*

Proof. Suppose $l(s) = l_N$. We consider only the case of $t(N) = (e, i, j, 1)$ since another case is treated quite similarly. We assume that the requirement $R(e, i, j, 1)$ is never certified after stage s and deduce a contradiction. Since $a_i \not\leq b_j$, we see that $A_i \not\leq_{\mathrm{T}}^{\mathrm{hp}} A_{G(j)}$. Thus, there is an x with $p_e(l_N) \leqslant |x|$ such that $A_i(x) \neq \Phi_e(A_{G(j)}, x)$. Take the least such x, and let n be such that $\sigma_n = A_{G(j)} \upharpoonright p_e^x$. Suppose, for example, that $A_i(x) = 1$ and $\Phi_e(A_{G(j)}, x) = 0$. Take a sufficiently large $s_0 > s$ so that $|x| \leqslant s_0$ and

$$(\forall t \geqslant s_0)\ [A_{i,t}(x) = A_i(x)\ \&\ A_{G(j),t} \upharpoonright p_e^x = A_{G(j)} \upharpoonright p_e^x].$$

Then since (1) of case 1.1.1 occurs at each stage $t+1$ after s_0, $\langle x, n\rangle$ is eventually put into $V(e, i, j, 1)$. Thus, we have that $\langle x, n\rangle \in V(e, i, j, 1) = W_{\theta(e,i,j,1)}$, $x \in A_i$ and $\sigma_n = A_{G(j)} \upharpoonright p_e^x$. It follows that $\theta(e, i, j, 1) \in H_{i,G(j)}$ by the definition of $H_{i,G(j)}$. On the other hand, since we are assuming $R(e, i, j, 1)$ is never certified after s, case 1.1.1.1 does not occur, which implies that $h_{i,G(j)}(\theta(e, i, j, 1), t+1) = 0$ for all $t \geqslant s_0$. By taking the limit, we have $\theta(e, i, j, 1) \notin H_{i,G(j)}$. This is a contradiction. $\square$

From the claim, we see that the sequence $\{l_N\}_N$ is well defined.

Claim 4.2. *Each requirement is attacked only finitely often.*

Proof. Suppose $R(e, i, j, k)$ is the requirement with the highest priority which is attacked infinitely often, say $k = 1$. Another case is the same. Then there is an s_0 such that the requirements with higher priority than $R(e, i, j, 1)$ are never attacked after s_0. Take a sufficiently large $s_1 \geqslant s_0$, so that

$$\begin{aligned}(\forall s \geqslant s_1)\ [&h_{i,G(j)}(\theta(e, i, j, 1), s) = H_{i,G(j)}(\theta(e, i, j, 1))\\ &\&\ \hat{h}_{i,G(j)}(\hat{\theta}(e, i, j, 1), s) = \hat{H}_{i,G(j)}(\hat{\theta}(e, i, j, 1))].\end{aligned}$$

There must be an $s_2 \geqslant s_1$ such that $R(e, i, j, 1)$ is injured at s_2 Then, we have $h_{i,G(j)}(\theta(e, i, j, 1), s_2) = \hat{h}_{i,G(j)}(\hat{\theta}(e, i, j, 1), s_2) = 0$ (see case 3.1), and $R(e, i, j, 1)$ is the first member of *LIST*. Thus, $R(e, i, j, 1)$ is attacked at stage $s_2 + 1$. By Claim 4.1, it is certified at some stage s_3 such that $s_3 > s_2 + 1$. Then we have $h_{i,G(j)}(\theta(e, i, j, 1), s_3) = 1$ or $\hat{h}_{i,G(j)}(\hat{\theta}(e, i, j, 1), s_3) = 1$, which is obviously a contradiction. $\square$

Claim 4.3. *Every requirement is satisfied.*

Proof. We show that the requirement $R(e, i, j, 1)$ is satisfied. Another case is the same. First we prove that it must be the case that either $H_{i,G(j)}(\theta(e, i, j, 1)) = 1$

or $\hat{H}_{i,G(j)}(\hat{\theta}(e,i,j,1))=1$. Suppose, on the contrary, that $H_{i,G(j)}(\theta(e,i,j,1))=\hat{H}_{i,G(j)}(\hat{\theta}(e,i,j,1))=0$. Then we have

$$(\forall\langle x,n\rangle\in V(e,i,j,1))\ [A_i(x)=0\ \vee\ \sigma_n\neq A_{G(j)}\upharpoonright p_e^x],$$

and

$$(\forall\langle x,n\rangle\in\hat{V}(e,i,j,1))\ [A_i(x)=1\ \vee\ \sigma_n\neq A_{G(j)}\upharpoonright p_e^x].$$

Take a sufficiently large s_0 so that

$$(\forall s\geqslant s_0)\ [h_{i,G(j)}(\theta(e,i,j,1),s)=H_{i,G(j)}(\theta(e,i,j,1))$$
$$\&\ \hat{h}_{i,G(j)}(\hat{\theta}(e,i,j,1),s)=\hat{H}_{i,G(j)}(\hat{\theta}(e,i,j,1))].$$

By Claim 4.2, there is an $s_1\geqslant s_0$ such that the requirements with priority higher than or equal to $R(e,i,j,1)$ are never attacked after s_1. Since $R(e,i,j,1)$ is never attacked, $V_s(e,i,j,1)$ and $\hat{V}_s(e,i,j,1)$ are fixed, so they are finite. Thus, there is an $s_2\geqslant s_1$ such that, for every $t\geqslant s_2$,

$$(\forall\langle x,n\rangle\in V_t(e,i,j,1)\cup\hat{V}_t(e,i,j,1))\ [A_{i,t}(x)=A_i(x)\ \&\ A_{G(j),t}\upharpoonright p_e^x=A_{G(j)}\upharpoonright p_e^x].$$

It follows that, for every $t\geqslant s_2$, $h_{i,G(j)}(\theta(e,i,j,1),t)=\hat{h}_{i,G(j)}(\hat{\theta}(e,i,j,1),t)=0$ and

$$(\forall\langle x,n\rangle\in V_t(e,i,j,1))\ [A_{i,t}(x)=0\ \vee\ \sigma_n\neq A_{G(j),t}\upharpoonright p_e^x],$$

and

$$(\forall\langle x,n\rangle\in\hat{V}_t(e,i,j,1))\ [A_{i,t}(x)=1\ \vee\ \sigma_n\neq A_{G(j),t}\upharpoonright p_e^x].$$

These together imply that $R(e,i,j,1)$ is injured at any stage t after s_2, which contradicts Claim 4.2.

Now suppose $H_{i,G(j)}(\theta(e,i,j,1))=1$, and let $\langle x,n\rangle$ be an element of $V(e,i,j,1)$ such that $A_i(x)=1$ and $\sigma_n=A_{G(j)}\upharpoonright p_e^x$. Suppose $\langle x,n\rangle\in V_{s+1}(e,i,j,1)-V_s(e,i,j,1)$. Then $A_{i,s+1}(x)=1$, $\Phi_e(A_{G(j),s+1},x)=0$ and $\sigma_n=A_{G(j),s+1}\upharpoonright p_e^x$. It follows that $\Phi_e(A_{G(j)},x)=0$ since $A_{G(j)}\upharpoonright p_e^x=A_{G(j),s+1}\upharpoonright p_e^x=\sigma_n$. By the definition of C_k, $B_j=A_{G(j)}$ on p_e^x, which implies that $A_i(x)\neq\Phi_e(B_j,x)$ as required.

Similarly, in the case of $\hat{H}_{i,G(j)}(\hat{\theta}(e,i,j,1))=1$, we also obtain that $A_i(x)\neq\Phi_e(B_j,x)$. $\square$

5. Remarks and problems

Ladner [9] proved that the p-T degrees of recursive sets are dense. For nonrecursive sets, Homer [6, 7] showed that there is no minimal p-T degree: if A is not recursive, then the set $\{x0^{2^{|x|}}: x\in A\}$ has the p-T degree between $\deg_{\mathrm{T}}^{\mathrm{p}}(\emptyset)$ and $\deg_{\mathrm{T}}^{\mathrm{p}}(A)$. He left open the possibility that there are sets A and B with $A<_{\mathrm{T}}^{\mathrm{p}}B$ such that there is no p-T degree between $\deg_{\mathrm{T}}^{\mathrm{p}}(A)$ and $\deg_{\mathrm{T}}^{\mathrm{p}}(B)$. Downey [5] also asked whether any p-T degree has a minimal cover. We remark that Ladner's proof of the density of $\boldsymbol{R}_{\mathrm{T}}^{\mathrm{p}}$ can be applied to nonrecursive sets and that the p-T degrees of all sets are dense. This answers Downey's question negatively.

Proposition 5.1. *For any A, B with $A <_T^p B$, there is a set C such that $A <_T^p C <_T^p B$.*

Proof. Suppose $A <_T^p B$. We may assume that $A = B \cap 0\Sigma^*$ since $A \oplus B \equiv_T^p B$. We define an increasing sequence $\{l_s\}_s$ and a sequence $\{C_s\}_s$ of finite functions by recursion, then we set $C = \bigcup_s C_s$.

We set $l_0 = 0$ and $C_0 = \emptyset$.

Suppose l_s and C_s are defined, and suppose $\mathrm{dom}(C_s) = \{z: |z| < l_s\}$.

Case 1: $s = 2e$. Let $C_s * A$ denote the function defined by: $C_s * A(z) = C_s(z)$ if $|z| < l_s$, and $C_s * A(z) = A(z)$ otherwise. Let Φ_e be the eth deterministic OTM with polynomial-time bound. Starting from the string 0^{l_s}, we successively compute $\Phi_e(C_s * A, x)$ and check whether $B(x) = \Phi_e(C_s * A, x)$ or not, where the queries are done to B during this process. Then we find the first x such that $|x| \geqslant l_s$ and $B(x) \neq \Phi_e(C_s * A, x)$ since we are assuming $A <_T^p B$. Let l_{s+1} be l_s plus the number of steps needed to find this x and verify the inequality $B(x) \neq \Phi_e(C_s * A, x)$. We extend C_s as follows. $\mathrm{dom}(C_{s+1}) = \{z: |z| < l_{s+1}\}$ and $C_{s+1}(z) = A(z)$ for $l_s \leqslant |z| < l_{s+1}$. Then we have the inequality $B \neq \Phi_e(C)$.

Case 2: $s = 2e+1$. Similarly, we search for the least x such that $|x| \geqslant l_s$ and $B(x) \neq \Phi_e(A, x)$, then define l_{s+1} as before. Let C_{s+1} be the extension of C_s defined by: $C_{s+1}(z) = C_s(z)$ if $|z| < l_s$ and $C_{s+1}(z) = B(z)$ if $l_s \leqslant |z| < l_{s+1}$. Then we see that $C \neq \Phi_e(A)$.

The sequence $\{l_s\}$ is recursive in B. To compute C from B in polynomial time, suppose x is given. We can find the unique s such that $l_s \leqslant |x| < l_{s+1}$ by performing the construction in $|x|$ steps. If $s = 2e$ then $C(x) = A(x)$, and if $s = 2e+1$ then $C(x) = B(x)$. Thus, we have $C \leqslant_T^p B$. $A \leqslant_T^p C$ since $A = B \cap 0\Sigma^* = C \cap 0\Sigma^*$. $\square$

The same argument is applied to extend some of the results on the p-T degrees of recursive sets to the p-T degrees of all sets. For example, we can show the combined splitting and density theorem (see [1, Corollary 4.4]) for the p-T degrees of arbitrary sets. However, it is not clear whether Corollary 4.10 of [1] holds for nonrecursive sets. Thus, we ask whether every p-T degree of nonrecursive set bounds a minimal pair.

Since the p-T degrees of all sets are dense as we remark above, it might be expected that the Π_2 theory of the p-T degrees of all sets coincides with that of $\boldsymbol{R}_T^p$. This has been left open in [11]. It is also not known whether the Π_2 theory of the hp-T degrees of the Δ_2^0 sets is decidable.

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